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DUALITY SYMMETRY OF STRING THEORY: A WORLDSHEET PERSPECTIVE

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Abstract

We study duality and local symmetries of closed bosonic string from the perspectives of worldsheet approach in the phase space path integral formalism. It is shown that the Ward identities reflecting the local symmetries associated with massless excitations such as graviton and antisymmetric tensor can be cast in a duality covariant form. It is shown how the manifestly $O(d,d)$ invariant Hamiltonian can be obtained in the Hassan-Sen toroidal compactification scheme, d being the number of compact dimensions. It is proposed that massive excited states possess a T-duality symmetry for constant (tensor) backgrounds. This conjecture is verified for the first massive level.

One of the marvels of the string theory is its rich symmetry contents and notable among these are the dualities. The underlying string dynamics in diverse dimensions is primarily understood through the web of dualities which unravel intimate connections between different string theories. It is recognized that target space duality, the T-duality, can be tested in perturbation theory. When we consider evolution of a string in the background of its massless excitation in the first quantized approach, the worldsheet action is expressed as a 2-dimensional σ-model action and the massless backgrounds play the role of coupling constants. The vanishing of the corresponding β-functions lead to the "equations of motion" of the those backgrounds. The string effective actions have played very important role in understanding of string theory from several perspectives. Moreover, if we adopt toroidal compactification and require that the backgrounds do not depend on these compact coordinates, then the reduced effective action manifests the associated T-duality symmetry. The target space duality is also understood from the worldsheet point of view. In this approach, we associate a dual coordinate for every compact direction of the string coordinate and derive equations of motion for each of the set. Furthermore, with suitable combination of the two sets, equations of motion can be expressed in a manifestly duality covariant form.

The string effective action is known to be invariant under target space local symmetries such as general coordinate transformation, associated with the graviton, vector gauge transformation, associated with the two-form antisymmetric field and nonabelian gauge transformations in the presence of nonabelian massless gauge fields which appear in certain compactified theories. There is a proposal to unravel these local symmetries from the worldsheet view point.

The purpose of this letter is two fold. It is argued that, at least for closed bosonic string, the Hamiltonian description manifestly exhibits the duality symmetries. Furthermore, we derive Ward identities intimately related to the symmetries of the afore mentioned massless states of the string which are covariant under duality transformations. We adopt the phase space Hamiltonian formalism to derive these results. These will be stated more precisely in sequel. Furthermore, we present some evidence that excited massive string states also exhibit duality symmetry. These are similar to $R \leftrightarrow \frac{1}{R}$ duality symmetry. At this stage, we can verify our conjecture when the higher dimensional backgrounds (tensors) are constant.

Let us consider a closed bosonic string in the background of its massless excitations graviton, G_{MN}, and antisymmetric tensor B_{MN}, where target spacetime indices, $M, N = 1, 2, ...\mathcal{D}$.

$$S = \frac{1}{2}\int d\sigma d\tau \Big(\gamma^{ab}\sqrt{-\gamma}G_{MN}(X)\partial_a X^M \partial_b X^N + \epsilon^{ab} B_{MN}(X)\partial_a X^M \partial_b X^N\Big) \qquad (1)$$

Here $X^M(\sigma,\tau)$ are string coordinates and γ^{ab} is the worldsheet metric. The classical action is invariant under worldsheet coordinate reparametrization. A simple example of worldsheet duality symmetry is to consider flat target space metric and set

$B_{MN}=0$. The spectrum is invariant under $\sigma \leftrightarrow \tau$ which amounts to $P_M \leftrightarrow X'^M$, P_M being the canonical momenta, prime and 'overdot', denote derivatives with respect to σ and τ respectively. Moreover, if one compactifies a spatial coordinate of a closed string on S^1 with radius R, the perturbative spectrum matches with that of another string if the corresponding coordinate is compactified on a circle of radius $\frac{1}{R}$ when we interchange the Kaluza-Klein modes with the winding modes and $R \leftrightarrow \frac{1}{R}$; subsequently this symmetry has been studied in more general settings [1, 2]. When some of the spatial coordinates of a string are compactified on torus, T^d, d being the number of compact directions with constant backgrounds $G_{\alpha\beta}$ and $B_{\alpha\beta},\ \alpha,\beta=1,2,..d$, the duality group is $O(d,d,\mathbf{Z})$, $\mathbf{Z}$ being integers. If the backgrounds assume only time dependence, the string effective action is expressed in a manifestly $O(D,D)$ invariant form, where D is the number of spatial dimensions [12] which has interesting consequences in string cosmology [4, 2]. In a more generalized setting one adopt a toroidal compactification scheme when the target space manifold $\mathcal{S}$ is decomposed to $\mathcal{S}=S_{spacetime}\otimes K$ where $D=0,1,..D-1$ are the spatial dimensions and $K=T^d$ with $D+d=\mathcal{D}$. Furthermore, if the backgrounds $g_{\mu\nu},\ b_{\mu\nu},\ \mu,\nu=0,1,..D-1$ and $G_{\alpha\beta},\ B_{\alpha\beta},\ \alpha,\beta=1,2,..d$ depend only on the spacetime coordinates x^μ, then the reduced effective action is expressed in a manifestly $O(d,d)$ invariant form [5]. It is worth while to recall some of the salient features of T-duality from the worldsheet perspective. It was shown by Duff [6], for constant backgrounds G and B, that the evolution equations of the string coordinates can be cast in an $O(\mathcal{D},\mathcal{D})$ covariant form. For each string coordinate X^M, he introduced a dual set of coordinates $\tilde{Y}^M$ and expressed the equations of motion of the $2\mathcal{D}$ coordinates in the duality covariant form. For the next simplest case, if G and B assume time dependence it was shown that the worldsheet equation of motion can be expressed in an $O(\mathbf{D},\mathbf{D})$ covariant form where $\mathbf{D}$ are the number of spatial dimensions [7]. On this occasion, for each spatial string coordinate, X^I, I being the spatial index, a dual coordinate $\tilde{Y}^I$ was introduced and combined equations were cast in manifestly 'duality' covariant form. The worldsheet approach to T-duality for toroidal compactification was addressed by Schwarz and JM [5] in a general frame work and it was demonstrated that by, introducing dual coordinates along compact dimensions, an $O(d,d)$ covariant worldsheet equations of motion can be derived. Subsequently, Siegel has advanced these ideas in another direction, introducing the two vierbein formalism and extending them to supersymmetric theories [8].

Let us briefly recapitulate essentials of phase space Hamiltonian formalism and reformulate the problem in a duality invariant frame work. The two constraints associated with τ and σ reparametrization respectively are

$$\begin{aligned} \mathcal{H}_c &= \frac{1}{2}\Big(P_M P_N G^{MN} + X'^M X'^N G_{MN} - P_M G^{MP} B_{PN} X'^N \\ &+ X'^M B_{MP} G^{PN} P_N - B_{MP} G^{PQ} B_{QN} X'^M X'^N\Big) \simeq 0 \\ & P_M X'^M \simeq 0 \end{aligned} \tag{2}$$

$\mathcal{H}_c$ is the canonical Hamiltonian derived from (1) These are primary constraints which vanish weakly, derived without any specific choice of the worldsheet metric, γ^{ab}. In order to express them in a duality invariant form, let us combine P_M and X'^M to define a $\mathcal{D}$-dimensional $O(\mathcal{D},\mathcal{D})$ vector

$$\mathcal{V} = \begin{pmatrix} P_M \\ X'^M \end{pmatrix} \tag{3}$$

The canonical Hamiltonian density can be re-expressed as

$$\mathcal{H}_c = \frac{1}{2}\mathcal{V}^T \mathcal{M} \mathcal{V} \tag{4}$$

in a matrix notation where $\mathcal{M}$ is a $2\mathcal{D} \times 2\mathcal{D}$ matrix [9]

$$\mathcal{M} = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} \tag{5}$$

where G and B stand for backgrounds $G_{MN}(X)$ and $B_{MN}(X)$ appearing in (1). Note that the equal τ canonical Poisson Bracket (PB) relation

$$\{X^M(\sigma), P_N(\sigma')\}_{PB} = \delta^M_N \delta(\sigma - \sigma') \tag{6}$$

translates to

$$\{\mathcal{V}(\sigma), \mathcal{V}(\sigma')\}_{PB} = \eta \frac{d}{d\sigma'}\delta(\sigma - \sigma') \tag{7}$$

where $\eta = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$, the $2\mathcal{D} \times 2\mathcal{D}$ matrix is the $O(\mathcal{D},\mathcal{D})$ metric, $\mathbf{1}$ being the $\mathcal{D} \times \mathcal{D}$ unit matrix. Under the global $O(\mathcal{D},\mathcal{D})$ transformations

$$\mathcal{M} \to \Omega\mathcal{M}\Omega^T, \quad \Omega^T\eta\Omega = \eta, \quad \Omega \in O(\mathcal{D},\mathcal{D}) \tag{8}$$

The linear combinations of the above two constraints (2)

$$\mathcal{L}_\pm = \frac{1}{2}\mathcal{H}_c \pm \frac{1}{4}\mathcal{V}^T\eta\mathcal{V} \tag{9}$$

satisfy the equal τ PB algebra

$$\{\mathcal{L}(\sigma)_\pm, \mathcal{L}(\sigma')_\pm\}_{PB} \simeq \pm\Big(\mathcal{L}(\sigma)_\pm + \mathcal{L}(\sigma')_\pm\Big)\frac{d}{d\sigma}\delta(\sigma - \sigma') \tag{10}$$

and

$$\{\mathcal{L}_+(\sigma), \mathcal{L}_-(\sigma')\}_{PB} = 0 \tag{11}$$

Therefore, $\mathcal{L}_\pm(\sigma)$ are a pair of first class constraints and the theory is covariantly quantized adopting Fradkin-Vilkovisky Hamiltonian formalism [10]. In the context of closed string, in the background of its massless excitations, the Hamiltonian phase space BRST quantization was carried out by us [11, 12]. The corresponding BRST charge is obtained by adopting the standard procedure

$$\mathcal{Q}_{BRST} = \int d\sigma \Big[\mathcal{L}_+\eta_+ + \mathcal{L}_-\eta_- + \mathcal{P}_+\eta_+\eta'_+ - \mathcal{P}_-\eta_-\eta'_-\Big] \tag{12}$$

Here the pair of ghosts $\{\eta_+, \eta_-\}$ are introduced, as is the prescription, for the two first class constraints $\{\mathcal{L}_+, \mathcal{L}_-\}$ which depend on $\mathcal{V}$ and the backgrounds, $\mathcal{M}$. $\{\mathcal{P}_+, \mathcal{P}_-\}$ are conjugate ghost momenta. The gauge fixed Hamiltonian density $\mathcal{H}_\zeta = \{\zeta, \mathcal{Q}_{BRST}\}_{PB}$. For choice of orthonormal gauge: $\zeta = \mathcal{P}_+ + \mathcal{P}_-$ and

$$\mathcal{H}_{ON} = \mathcal{L}_+ + \mathcal{L}_- + 2\mathcal{P}_+\eta'_+ + \mathcal{P}'_+\eta_+ - 2\mathcal{P}_-\eta'_- - \mathcal{P}'_-\eta_- \tag{13}$$

This was the starting point to derive Ward Identities (WI) associated with the massless states of the closed string. As alluded to above, we intend to obtain similar WI in a duality covariant manner. The first step is to introduce the Hamiltonian action

$$S_H = \int d\sigma d\tau \Big[P_M \dot{X}^M - \mathcal{H}_{ON}\Big] \tag{14}$$

In order to unravel the symmetry encoded due to general coordinate transformation invariance which is intimately related to the presence of graviton, let us consider a generating functional

$$\mathcal{Q}_G = \int d\sigma P_M \xi^M(X(\sigma)) \tag{15}$$

responsible for an infinitesimal transformation, $\xi^M(X)$ being the parameter. The variations of phase space variables, ghosts and the $O(\mathcal{D},\mathcal{D})$ vectors are obtained by evaluating their PB with $\mathcal{Q}_G$ i.e.

$$\delta_{\mathcal{Q}_G}\mathcal{V} = \{\mathcal{V}, \mathcal{Q}_G\}_{PB}, \quad \delta_{\mathcal{Q}_G}\eta_\pm = 0, \quad \delta_{\mathcal{Q}_G}\mathcal{P}_\pm = 0 \tag{16}$$

and in particular $\delta X^M = \xi^M(X)$; indeed $\mathcal{Q}_G$ induces general coordinate transformations. Since the arguments of G_{MN} and B_{MN} are shifted their variations under (15) are

$$\begin{aligned} \delta_{\mathcal{Q}_G} G_{MN}(X) &= G_{MN},_P(X)\xi^P(X), \qquad \delta_{\mathcal{Q}_G} G^{MN}(X) = G^{MN},_P(X)\xi^P(X), \\ & \delta_{\mathcal{Q}_G} B_{MN}(X) = B_{MN},_P(X)\xi^P(X) \end{aligned} \tag{17}$$

comma stands for the ordinary derivative here and everywhere. Thus the components, $\mathcal{M}^{MN}, \mathcal{M}^M_N$ and M_{MN}, of the $\mathcal{M}$-matrix being functions of X also transform according to the above prescriptions. The variation of the action is

$$\delta_{\mathcal{Q}_G} S_H \sim \int d\sigma \Big[\frac{1}{2}\delta_{\mathcal{Q}_G}\mathcal{V}^T\mathcal{M}\mathcal{V} + \frac{1}{2}\mathcal{V}^T\delta_{\mathcal{Q}_G}\mathcal{M}\mathcal{V} + \frac{1}{2}\mathcal{V}^T\mathcal{M}\delta_{\mathcal{Q}_G}\mathcal{V}\Big] \tag{18}$$

It is a straight forward and tedious calculation to check that

$$\delta_{\mathcal{Q}_G} S_H = -\delta_{GCT} S_H \tag{19}$$

The *r.h.s.* of the above equation is to be interpreted as follows. The Hamiltonian action, S_H depends on $\mathcal{M}$, expressed in terms of G_{MN}, G^{MN}, and B_{MN}. These tensors transform according to the rules given below [13]

$$\begin{aligned} \delta_{GCT} G_{MN} &= -G_{MP}\xi^P{}_{,N} - G_{PN}\xi^P{}_{,M} - G_{MN,P}\,\xi^P \\ \delta_{GCT} B_{MN} &= -B_{MP}\xi^P{}_{,N} - B_{PN}\xi^P{}_{,M} - B_{MN,P}\,\xi^P \end{aligned} \tag{20}$$

The next step is to define the Fradkin-Tseytlin generating functional, Σ, in the phase space Hamiltonian path integral formalism [14]

$$\Sigma[\mathcal{M}] = \int \mathcal{D}P\mathcal{D}X\mathcal{D}\eta_\pm \mathcal{D}\mathcal{P}_\pm e^{iS_H[P,X',\eta_\pm,\mathcal{P}_\pm,\mathcal{M}]} \tag{21}$$

Notice that under canonical transformations, the phase space measure is invariant, at least classically. The issue of noninvariance of this measure, which might lead to anomalies, will be touched upon briefly later. Moreover, if we implement the canonical transformation (15) on $\Sigma[\mathcal{M}]$ and the variation of S_H under (15) is compensated through (19), as was argued in [11, 12]. Then we arrive at

$$\delta_{GCT}\Sigma[\mathcal{M}] = \Big\langle \int d^D x \frac{\delta S_H}{\delta \mathcal{M}(x)} \delta\mathcal{M}(x) \Big\rangle_{\mathcal{M}} = 0 \tag{22}$$

In the above equation $< ... >$ is to be understood as the functional integral weighed with $exp(-iS_H)$. Note that the functional derivative of the action, S_H, with respect to the background is the corresponding vertex operator. Therefore, (22) translates to

$$\Big\langle \int d^2\sigma \delta(x - X(\sigma)) V^{PN}_{\mathcal{M}} \Big(\mathcal{M}_{PR}\xi^R{}_{,N} + \mathcal{M}_{RN}\xi^R{}_{,P} + \mathcal{M}_{PN,R}\,\xi^R \Big) \Big\rangle_{\mathcal{M}} = 0 \tag{23}$$

where $V^{PN}_{\mathcal{M}} = \frac{\delta S_H}{\delta \mathcal{M}_{PN}}$. It is understood that $\mathcal{M}$ has contravariant, covariant and mixed indices. Therefore, rules for GCT should be adopted accordingly [13]. In order to verify that the $\mathcal{M}$ derivative of S_H reproduces the vertex operator; one explicit check is that, for a simple case when we have G_{MN} as the only background and set it to the flat space metric after taking the functional derivatives of S_H with respect to G_{MN}. Then we reproduce the graviton vertex operator.

Now we are ready to derive the WI. Note that the infinitesimal parameter, $\xi^M(X)$ is arbitrary. Therefore, we may functionally differentiate (23) with respect to $\xi^M(X)$ and then set $\xi^M = 0$. Subsequently, let us take functional derivatives of the resulting expression with respect to the backgrounds $\mathcal{M}_{P_iQ_i}(y_i)$, $\{y_i\}$ are the spacetime coordinates, and examine the consequences

$$\begin{aligned} &\Pi^n_{i=1} \frac{\delta}{\delta \mathcal{M}_{P_iQ_i}} \Big\langle \int d^2\sigma V^{PN}_{\mathcal{M}} \Big[\mathcal{M}_{PQ}\partial_N \delta(x - X) + \mathcal{M}_{QN}\partial_P \delta(x - X) \\ &+ \mathcal{M}_{PN,Q}\, \delta(x - X) \Big] \Big\rangle_{\mathcal{M}} = 0 \end{aligned} \tag{24}$$

It is understood that at the end of the operations the backgrounds are to set the required configurations which is the meaning of $< .... >_{\mathcal{M}}$ in eq.(22) - eq.(24). These are desired WI which involve the massless states. Let us analyze (24) more carefully. The $\mathcal{M}$-derivatives act in three ways: (i) When it operates on $< ... >$ action of each derivative brings down a vertex operator $\int d^2\sigma V_{\mathcal{M}}^{P_iQ_i}\delta(y_i - X(\sigma))$ due to the presence of the measure e^{-iS_H} in the definition of $< ... >$. Thus we have eventually $(n+1)$-vertex operators after n-operations. (ii) When a derivative acts on the vertex operator, $V_{\mathcal{M}}^{PN}(X(\sigma))$ it will kill any $\mathcal{M}$-dependence in the vertex operator and a corresponding δ-function will appear. (iii) The derivatives also act on the $\mathcal{M}$-terms appearing in (24) in the square bracket with the δ-functions and their derivatives. Recall that $\mathcal{M}$ is an O(d,d) matrix and it must be kept in mind while taking functional derivatives. In order to make it more transparent, if we consider the above expression in the momentum representation, we notice that $(n+1)$-point function contracted with momentum can be expressed in terms of linear combination of lower point functions (with contact terms due to the presence of δ-functions). Notice that the WI is $O(\mathcal{D},\mathcal{D})$ covariant. Moreover, adopting the canonical transformation introduced in [12], the gauge symmetry associated with the 2-form field B_{MN} can be revealed, if we choose $\mathcal{Q}_\Lambda = \int d\sigma X'^M\Lambda_M$. Indeed, as has been noted by Siegel [8], if we define

$$\mathcal{W} = \begin{pmatrix} \xi^M \\ \Lambda_M \end{pmatrix} \tag{25}$$

as an $O(\mathcal{D},\mathcal{D})$ vector then we can construct charges (generating functionals for canonical transformations) as follows:

$$\mathcal{Q}_G + \mathcal{Q}_B = \int d\sigma \mathcal{W}^T\mathcal{V} \tag{26}$$

One can check that operation $\mathcal{Q}_G + \mathcal{Q}_B$ on S_H gives us a relation

$$(\delta_{Q_G} + \delta_{Q_B})S_H = -\delta_{GCT}S_H - \delta_{Gauge}S_H \tag{27}$$

where the second transformation on the $r.h.s$ is interpreted to be gauge variation of background B_{MN} in $\mathcal{M}$-matrix as

$$\delta_{Gauge}B_{MN} = \partial_M\Lambda_N - \partial_N\Lambda_M \tag{28}$$

$\Lambda_M(X)$ being the vector gauge parameter associated with B_{MN}. Thus, we can derive combined WI, starting from $\Sigma(\mathcal{M})$ and use (25), which in its full form will be manifestly duality covariant.

We now consider compactification of the closed string on d-torii, T^d, when the backgrounds along compact directions are independent of those coordinates, depend only on noncompact coordinates, $X^\mu(\sigma\tau)$, $\mu = 1,2..D-1$ and the compact coordinates are $Y^\alpha(\sigma\tau)$, $\alpha = 1,2,..d$ with $D + d = \mathcal{D}$. We adopt the Hassan-Sen [15, 8] compactification scheme where the backgrounds, G_{MN} and B_{MN}, are decomposed into following block diagonal forms

$$G_{MN}(X) = \begin{pmatrix} g_{\mu\nu}(X(\sigma)) & 0 \\ 0 & G_{\alpha\beta}(X(\sigma)) \end{pmatrix}, \quad B_{MN} = \begin{pmatrix} b_{\mu\nu}(X(\sigma)) & 0 \\ 0 & B_{\alpha\beta}(X(\sigma)) \end{pmatrix} \tag{29}$$

Thus the action (1) can be decomposed into two parts as evident from (29). The corresponding canonical Hamiltonian density is

$$\mathcal{H}_c = \frac{1}{2}\Big(\mathcal{V}_1^T \mathbf{M} \mathcal{V}_1 + \mathcal{V}_2^T \tilde{\mathbf{M}} \mathcal{V}_2\Big) \tag{30}$$

which can be written as $\mathcal{H}_c = \mathcal{H}_1 + \mathcal{H}_2$, the first term being $\mathcal{H}_1$ and $\mathcal{H}_2$ is the second one eq.(30); the two vectors being

$$\mathcal{V}_1 = \begin{pmatrix} P_\mu \\ X'^\mu \end{pmatrix}, \quad \mathcal{V}_2 = \begin{pmatrix} \tilde{P}_\alpha \\ Y'^\alpha \end{pmatrix} \tag{31}$$

The matrices $\mathbf{M}$ and $\tilde{\mathbf{M}}$ are defined to be

$$\mathbf{M} = \begin{pmatrix} g^{\mu\nu} & -g^{\mu\rho} b_{\rho\nu} \\ b_{\mu\rho} g^{\rho\nu} & g_{\mu\nu} - b_{\mu\rho} g^{\rho\lambda} b_{\lambda\nu} \end{pmatrix}, \quad \tilde{\mathbf{M}} = \begin{pmatrix} G^{\alpha\beta} & -G^{\alpha\gamma} B_{\gamma\beta} \\ B_{\alpha\gamma} G^{\gamma\beta} & G_{\alpha\beta} - B_{\alpha\gamma} G^{\gamma\delta} B_{\delta\beta} \end{pmatrix} \tag{32}$$

Notice that $\mathbf{M}$ and $\tilde{\mathbf{M}}$ are $O(D,D)$ and $O(d,d)$ matrices respectively and depend on the noncompact coordinate $X(\sigma)$. The pair $\mathcal{V}_1$ and $\mathcal{V}_2$ are corresponding two vectors of $O(D,D)$ and $O(d,d)$. The global $O(D,D)$ transformation is implemented by the Ω_1-matrices and Ω_2 implements the $O(d,d)$ transformation satisfying the properties analogous to eq.(8); their corresponding metrics are

$$\eta_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{33}$$

whereas η_1 is a $2D \times 2D$ matrix, η_2 is a $2d \times 2d$ matrix. Thus the gravitational WI originating from $\mathcal{H}_1$ can be derived following the procedure given above. However, there are massless scalars(moduli), $G_{\alpha\beta}$ and $B_{\alpha\beta}$ which appear in $\mathcal{H}_2$. These depend on X^μ and therefore, under canonical transformation (15) they transform accordingly. The technique of [16] can be appropriately used to derive the gravitational WI. Similarly, the gauge WI associated with the 2-form field could be obtained in a straight forward manner. Another important conclusion is that the canonical Hamiltonian density,$\mathcal{H}_c$, is invariant under global $O(d,d)$ transformation since

$$\tilde{\mathbf{M}} \to \Omega_2 \tilde{\mathbf{M}} \Omega_2^T, \quad \mathcal{V}_2 \to \Omega_2 \mathcal{V}_2 \tag{34}$$

leaving $\mathcal{H}_2$ invariant whereas $\mathcal{H}_1$ is inert under $O(d,d)$ transformation. Furthermore, if we look at the Hamilton's equations of motion associated with the compact coordinates and their conjugate momenta, $\{Y^\alpha, P_\alpha\}$, we notice that these are conservation laws (since backgrounds $G_{\alpha\beta}$ and $B_{\alpha\beta}$ depend only on X^μ) and the resulting equations

of motion will be $O(d,d)$ covariant. Thus we find that the phase space Hamiltonian approach transparently exposes the duality symmetry.

It is worth while to discuss a few more issues relevant to present investigation. How can we derive the "equations of motion" of the background fields in this frame work? It can be achieved by resorting to an elegant and efficient technique proposed in [18] to obtain the background equations of motion in Hamiltonian formalism. The quantum generators of conformal transformations were constructed by introducing a generating function technique. For the case at hand, the method of [18] can be suitably exploited if we express the $\mathcal{M}$-matrix in terms of generalized vielbeins adopted in [5]: $\mathcal{M} = \mathbf{V}^T\mathbf{V}$

$$\mathbf{V} = \begin{pmatrix} E^{-1} & -E^{-1}B \\ 0 & E \end{pmatrix} \tag{35}$$

where E, the $\mathcal{D}\times\mathcal{D}$ matrix defines the metric $G_{MN} = E^TE$. We mention in passing that $\mathbf{V}\in O(\mathcal{D},\mathcal{D})$ since $\mathbf{V}^T\eta\mathbf{V} = \eta$. Thus the generators $\mathcal{L}_\pm$ will be expressed in terms of P_M, X'^M and $\mathbf{V}$. Now, following [18] we can compute anomalies in the quantum algebra of the generators. These will correspond to known equations of motion as derived earlier. More importantly, when we consider the case of compactified strings we note that the constraints obtained in terms of $\mathcal{H}_1$ and $\mathcal{H}_2$ will give equations of motion for $g_{\mu\nu}$ and $b_{\mu\nu}$ in terms of $\mathbf{M}$-matrix as well as for the $\tilde{\mathbf{M}}$-matrix. The Hamiltonian being $O(d,d)$-invariant the equations motion associated with the moduli is expected to be $O(d,d)$ covariant since we know that the dimensionally reduced effective action can expressed in $O(d,d)$ invariant form.

We have not discussed the dilaton coupling to the string so far. We recall that the dilaton couples to the ghosts and their conjugate momenta as was proposed in [12] adopting the arguments of [19]. Thus the full constraint algebra can be derived in the Hamiltonian framework and therefore, we can derive the equations of motion for all the massless background. The details of such calculations, in the present context, will presented in a separate publication.

Several remarks are in order in what follows. We have argued earlier that the phase space measure in the definition of $\Sigma[\mathcal{M}]$ is invariant under canonical transformations. However, when the transformed measure is carefully evaluated in the quantum theory, it might not be invariant signaling the appearance of an anomaly. We do not have a general prescription to check the presence of anomalies. In certain cases, the anomaly can be computed and with specific transformation prescriptions for the backgrounds it can be removed [20]. However, a general procedure to derive such anomalies is lacking in this worldshhet approach.

The Hamiltonian formalism treats the coordinates and their conjugate momenta on equal footing in the $2\mathcal{D}$-dimensional the phase space. The duality symmetry becomes quite transparent in the Hamiltonian descriptions from the worldsheet point of view. When we considered the Lagrangian formulation, the equations of motion could be cast in $O(d,d)$ covariant form provided one introduces dual coordinates for the com-

pact ones and the corresponding backgrounds are defined suitably in the dual space. In the past, it has been suggested that doubling of the number of coordinates might have underlying deep significance [6, 21]. The mathematical formulation of this approach is unquestionable; however, the physical significance of such theories are yet to be fully comprehended. Recently, some progress has been made to compute the β-functions in such a worldsheet approach [22]. Recently interests in the double field theory formulation have been revived due to a formulation in the target space [23] where the tensors G_{MN} and B_{MN} become functions of $2\mathcal{D}$ variables and the number of indices are also doubled. This is a consistent formulation of the new field theory and it has not found a direct application yet. Should one attempt a Hamiltonian formulation of the 'worldsheet double field theory' [22], the phase space will have twice the number of variables contrast to the conventional formulations and one will have to suitably define canonical variables in this frame work. It will be worth while to examine whether such theories are endowed with any enlarged symmetries.Note, however, that the $GL(D,R)$ symmetry introduced in [5] has been found to be important in the double field theory formulation.
It has been proposed that excited, massive stated might possess hitherto undiscovered symmetries [24, 25, 26, 27]. Moreover, some of the important properties of dual models, which are inherited by string theory, crucially depend on the fact that an infinite tower of states are exchanged in the scattering processes. Therefore, it is worth while to seek answer to the question whether the excited massive levels of a string exhibit any duality-like symmetry. If we examine the issue from the worldsheet view point, in the σ-model approach, the (massive) background coupling to the string is suppressed by mass term compared to coupling of massless states on purely dimensional considerations. Therefore, the duality symmetry we encounter, in study of the σ-model action in graviton and 2-form potential, will not be unraveled. Similarly, at the level of string effective action, the dimensionally reduced effective action exhibits duality symmetry (most commonly known $O(d,d)$) when we assume that the backgrounds do not depend on compact coordinates and thus ignore the KK modes. Thus the dualities associated with excited massive modes are to be envisaged from a different perspective. The evolution of string in its excited, massive backgrounds have been studied in the weak field approximation [26, 27]. One might assume, as a simple scenario, that the string is moving in the flat target space and the massive backgrounds are weak. Subsequently construct the vertex operators and demand them to be conformally invariant which already imposes strong constraints on them [24, 26]. As an illustrative example, consider a generic background coupling of closed bosonic string to its first massive level [26, 24]

$$F^{(1)}_{MNP}\partial X^M\partial X^N\bar{\partial}\bar{\partial}X^{P'},\ F^{(2)}_{MN'P'}\partial\partial X^P\bar{\partial}X^{N'}\bar{\partial}X^{P'},\ S_{\{MN\}\{P'Q'\}}\partial X^M\partial X^N\bar{\partial}X^{P'}\bar{\partial}X^{Q'} \tag{36}$$

The backgrounds $F^{(1)}, F^{(2)}$ and $F^{(3)}$ depend on string coordinates, X, and these term will be suppressed by factor of α' compared to the σ-model action for massless states on dimensional arguments. The vertex operator for the first excited massive level

will be sum of all such terms (call them vertex functions). These do not exhaust all possible vertex functions for the first massive level. The backgrounds fulfill gauge condition and satisfy equations of motion as a consequence of conformal invariance. If we require them to be $(1,1)$ primary these vertex functions are not independent; it is to be borne in mind that the stress energy tensors used to compute the weights are taken to be $T_{++} \sim \partial X \partial X$ and $T_{--} \sim \bar{\partial} X \bar{\partial} X$ in the flat target space.
In order to expose the conjectured duality, we resort to Hamiltonian description and assume that the tensors $F^{(i)}$ are spacetime independent as was first envisaged by Narain, Sarmadi and Witten for the heterotic string in constant graviton, antisymmetric tensor and gauge field backgrounds. Following pertinent points need attentions: (i) A careful reader will notice that, when $\{F^{(i)}\}$ are spacetime independent constant tensors, some of them or their linear combinations might be required to vanish once we demand that the vertex operator for the excited massive level be $(1,1)$ primary. However, all of them will not vanish. (ii) When we study T-duality symmetry from the worldsheet point of view in the presence of massless backgrounds, the resulting equations of motion are expressed in duality covariant form after incorporating the dual coordinates [5]. It was not essentials for those backgrounds (i.e. vertex operators) to be $(1,1)$ primary when we are seeking duality covariant equations of motion. Indeed, conformal invariance lets us decide which are the admissible background configurations. Therefore, in what follows, let us analyze how $P \leftrightarrow X'$ duality relates various (constant) tensor backgrounds. It will be obvious in the sequel that those tensors which will vanish on imposing $(1,1)$ primary conditions do not mix with the surviving ones under the duality transformations we are dealing with. Note that, in flat space $P_M = G^{(0)}_{MN}\dot{X}^N$ where $G^{(0)}_{MN} = \mathrm{diag}(+1,-1,-1...)$. In fact we express the vertex operators in terms of P_M, X'^M for our conveniences here and could replace $\partial X,\ \bar{\partial} X$ by $P \pm X'$ in above expressions as well. We would like to consider following vertex functions which can expressed as linear combination of appropriate F-tensors.

$$G^{(1)}_{MNQ}X'^M X'^N X''^Q, \quad G^{(2)MNQ}P_M P_N \dot{P}_Q, \quad G^{(3)Q}_{MN}X'^M X'^N \dot{P}_Q, \quad G^{(4)MN}_{Q}P_M P_N X''^Q,$$
$$G^{(5)}_{MNQR}X'^M X'^N X'^Q X'^R, \quad G^{(6)MNQR}P_M P_N P_Q P_R, \quad G^{(7)QR}_{MN}X'^M X'^N P_Q P_R \tag{37}$$

It is evident that one can construct more vertex functions for this level; however, it will suffice to deal with these six for the moment. Note that X'^M and P_M have the same dimensions as it true for the pair X''^M and $\dot{P}_M$. The simplest form of T-duality the interchange $\tau \leftrightarrow \sigma$ which implies $X'^M \leftrightarrow P_M$ and $X''^M \leftrightarrow \dot{P}_M$. If we desire that the interaction Hamiltonian consisting of sum of the six terms we have listed above, respect this duality symmetry, then following relations should hold

$$G^{(1)} \leftrightarrow G^{(2)}, \quad G^{(3)} \leftrightarrow G^{(4)}, \quad G^{(5)} \leftrightarrow G^{(6)} \tag{38}$$

and $G^{(7)}$ gets related to itself with appropriate shuffling of the indices. This transformation rule generalizes the interchange between G_{MN}, G^{MN} and B_{MN} for $X'^M \leftrightarrow P_M$ where $(G+B) \to (G+B)^{-1}$, alternatively the new metric $\mathcal{G}$ and the new 2-form $\mathcal{B}$

(all constants for us) are given by

$$\mathcal{G} = (G - BG^{-1}B), \quad \mathcal{B} = -G^{-1}B((G - BG^{-1}B) \tag{39}$$

Notice that these duality relations (38) hold amongst the (constant) background tensors of a given level. When we envisage the second massive excited level of the closed string there will be many more tensors; however that the vertex operator for the level (sum of all such vertex functions) will be suppressed by a factor α'^2 relative to the first massive level terms. One might seek answer to the question: Are there larger duality symmetries associated with massive levels beyond the discrete $P_M \leftrightarrow X'^M$ symmetry considered here?

To summarize: we have argued that the WI associated with the massless excitations of the closed string can be expressed in a duality covariant manner. It was accomplished by introducing generators of canonical transformations in the Hamiltonian phase space and defining the generating functional in path integral formalism. Furthermore, these generators [11, 12] can be combined to express in a duality invariant manner. The underlying local symmetries are manifest through the Ward identities. These WI's are to be treated as classical expression since anomalies might creep in; however, in certain cases it is possible to compute the anomalies and provide a prescription to remove them. We outlined a procedure to compute the quantum constraint algebra in order to derive the equations of motion for the backgrounds, $\mathcal{M}$, following the techniques introduced earlier in [18]. In fact if one adopts the proposal of Hohm, Hull and Zwiebach (HHZ) [23] to treat $\mathcal{M}$ as another $O(d,d)$ spacetime metric (in addition to η-matrix), then it might facilitate the computation of β-functions efficiently. However, it is to be kept in mind that HHZ's interpretation was in the context of double field theory. Therefore, whether truncation to (half) the spacetime variables will be useful or not is not obvious at this stage. We adopted Hassan-Sen compactification scheme and argued that WI can also be obtained for the massless moduli.

We have conjectured that there might be duality symmetries associated with each excited massive level of the closed string. We provided an example how the constant background tensors should transform to satisfy $P \leftrightarrow X'$ interchange. It argued that this type of duality will persist for higher excited states and the duality relation is to hold for each such level.

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References

1. K. Kikkawa and M. Yamazaki, Phys. Lett. **149B** (1984) 357;
   N. Sakai and I. Sanda, Prog. Theor. Phys. **75** (1986) 692;
   V. P. Nair, A Shapere, A. Strominger, and F. Wilczek, Nucl. Phys. **287B** (1987) 402;
   B. Sathiapalan, Phys. Rev. Lett. **58** (1987) 1597;
   R, Dijkgraaf, E. Verlinde, and H. Verlinde, Commun. Math. Phys. **115** (1988 649;
   K. S. Narain, Phys. Lett. **B169** (1986) 41;
   K. S. Narain, M. H. Sarmadi, and E. Witten, Nucl. Phys. **B279** (1987) 369;
   P. Ginsparg, Phys. Rev. **D35** (1987) 648;
   P. Gisnparg and C. Vafa, Nucl. Phys. **B289** (1987) 414;
   S. Cecotti, S. Ferrara and L. Giraldello, Nucl. Phys. **B308** (1988) 436;
   R. Brandenberger and C. Vafa, Nucl. Phys. **B316** (1988) 391;
   M. Dine, P.Huet, and N. Seiberg, Nucl. Phys. **B322** (1989) 301;
   J. Molera and B. Ovrut, Phys. Rev. **D40** (1989) 1146;
   G. Veneziano, Phys. Lett. **B265** 1991 287;
   A. A. Tseytlin and C. Vafa, Nucl. Phys. **B372** (1992) 443;
   M. Rocek and E. Verlinde, Nucl. Phys. **373** (1992) 630;
   J. H. Horne, G. T. Horowitz, and A. R. Steif, Phys. Rev. Lett. **68** (1992) 568;
   A.Sen, Phys. Lett. **B271** (1992) 295.
   E. Alvarez, L. Alvarez-Gaume, and Y. Lozano, Phys. Lett. **B336** (1994) 183.

2. For comprehensive discussions on dualities see the following three review articles: A. Giveon, M. Porrati and E. Rabinovici, Phys. Rep. **C244** 1994 77;
   J. E. Lidsey, D. Wands, and E. J. Copeland, Phys. Rep. **C337** 2000 343;
   M. Gasperini and G. Veneziano, Phys. Rep. **C373** 2003 1.

3. K. Meissner and G. Veneziano, Phys. Lett. **B267** (1991) 33; Mod. Phys. Lett. **A6** (1991) 3397.

4. M. Gasperini, J. Maharana, and G. Veneziano, Phys. Lett. **B272** 1991 277; Phys. Lett. **B296** 1992 51.

5. J. Maharana and J. H. Schwarz, Nucl. Phys. **B390** (1993) 3.

6. M. J. Duff, Nucl. Phys. **B335** 1990 610.

7. J. Maharana, Phys. Lett. **B296** (1992) 65. 43 (1996).

8. W. Siegel, Phys. Rev. **D47** (1993) 5453; Phys. Rev. **D48** (1993) 2826.

9. A. Shapere and F. Wilczek, Nucl. Phys. **B320** (1989) 669; A. Giveon, E. Rabinovici, and G. Veneziano, Nucl. Phys. **B322** (1989) 167; A. Giveon, N.

Malkin, and E. Rabinovici, Phys. Lett. **B220** (1989) 551; W. Lerche, D. Lüst, and N. P. Warner, Phys. Lett. **B231** (1989) 417.

10. E. S. Fradkin and G. A. Gilkovisky, Phys. Lett. **55B** (1975) 224;
M. Henneaux, Phys. Rep. **126C** (1986) 1;
S. Hwang, Phys. Rev. **D28** (1983) 2614.

11. G. Veneziano, Phys. Lett. **167B** (1985) 388.

12. J. Maharana and G. Veneziano, Nucl. Phys. **B283** (1987) 126; Phys. Lett. **169B** (1986) 177.

13. S. Weinberg Gravitation and Cosmology,John Wiley and Sons, New York (1972) p361, ; D. M. Capper and M Roman Medrano, Phys. Rev. **D9** (1974) 1641.

14. E. S. Fradkin and A. A. Tseytlin, Phys. Lett. **158B** (1985) 316; Nucl. Phys. **B261** (1985) 1.

15. S. F. Hassan and A. Sen, Nucl. Phys. **B375** (1992) 103.

16. J. Maharana, Phys. Lett. **B211** (1988) 431.

17. J. Maharana and J. H. Schwarz, Nucl. Phys. **B390**, 3 (1993).

18. S. Fubini, J. Maharana, M. Roncadeli, and G. Veneziano, Nucl. Phys. **B316** (1989) 36; also see A. Das, J. Maharana and S. Roy, Nucl. Phys. **329** (1990) 543.

19. T. Banks, D. Nemeshensky, and A. Sen, Nucl. Phys. **B277** (1986) 67.

20. A. Das, J. Maharana, and P. K. Panigrahi, Mod. Phys. Lett. **A3** (1988) 559.

21. E. Witten, Phys. Rev. Lett. **61** (1988) 670; A. A. Tseytlin, Phys. Lett. **B242** (1990) 163; Nucl. Phys. **B350** (1991) 395; Phys. Rev. Lett. **66** (1991) 545.

22. D. S. Berman and D. C. Thompson, Phys. Lett. **B662** (2008) 279. This paper contains more references on the subject.

23. T. Kugo and B. Zwiebach, Prog. Th. Phys. **87** (1992) 801; C. Hull and B. Zwiebach, JHEP, **0909** (2009) 099; C. Hull and B. Zwiebach, JHEP **0909** (2009) 090; A. Dabholkar and C. Hull, JHEP **0605** (2006) 009; O. Hohm, C. Hull and B. Zwiebach, arXiv:1006.4823 which has detailed references.

24. J. Maharana and G. Veneziano (unpublished works, 1986, 1991 and 1993); T. Kubota and G. Veneziano, Phys. Lett. **B207** (1988) 419.

25. J. Maharana, Novel Symmetries of String Theory, in String Theory and Fundamental Interactions, Springer Lecture Notes in Physics, Vol. **737** p525, Ed. G. Gasperini and J. Maharana Springer 2008, Berlin Heidelberg.

26. E. Evans and B. Ovrut, Phys. Rev. **D39** (1989) 3016; Phys. Rev. **D41** (1990) 3149.

27. R. Akhoury and Y. Okada; Nucl. Phys. **B318** (1989) 176.